1402.
2009.

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. a. Write down the formula of the linear Taylor's approximation of a function $f(x, y)$ near a point (x_0, y_0).
What can we say about the rate of this approximation?

Denote by $L^f(x, y)$ the linear function

$$L^f(x, y) := f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\,(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)\,(y - y_0).$$

Then

$$f(x, y) \approx L^f(x, y), \quad \text{near } (x_0, y_0),$$

in the sense that

$$\frac{|f(x, y) - L^f(x, y)|}{|\mathbf{X} - \mathbf{X}_0|} \to 0, \quad \text{as } |\mathbf{X} - \mathbf{X}_0| \to 0.$$

Here $\mathbf{X} = (x, y)$, $\mathbf{X}_0 = (x_0, y_0)$ and $|\mathbf{X} - \mathbf{X}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

b. Show that the tangent plane to the graph of a function $f(x, y)$ at a point $(x_0, y_0, z_0 = f(x_0, y_0))$ is a horizontal plane if and only if $\nabla f(x_0, y_0) = \mathbf{0}$.

The equation of the tangent plane is given by

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\,(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)\,(y - y_0).$$

Thus, $z = \text{const}$ is equivalent to $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$.

c. (For the function $f(x, y) = x + e^{xy}$ find the equation of its tangent plane at the point $(1, 0, 2)$.

We have,

$$\nabla f(x, y) = (1 + ye^{xy})\mathbf{i} + xe^{xy}\mathbf{j}.$$

Thus, $\nabla f(1, 0) = \mathbf{i} + \mathbf{j}$ so that the tangent plane is

$$z = 2 + (x - 1) + y = 1 + x + y.$$

d. For the function f from Part c., find a vector $\mathbf{u} \neq \mathbf{0}$ which is orthogonal to $\nabla f(1, 0)$.

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$,

$$\nabla f(1, 0) \cdot \mathbf{u} = u_1 + u_2.$$

Thus $\mathbf{u} = \mathbf{i} - \mathbf{j}$ is a possible choice.

2. a. (Let R be a region on the $xy-$ plane defined by

$$x^2 + y^2 \leq 1,\ x \geq 0,\ y \leq 0.$$

Find the integral

$$\iint_R e^{(x^2+y^2)} x^2 dxdy.$$

Use polar coordinates, r, ϕ. Then R is defined by $0 \leq r \leq 1$, $\frac{3\pi}{2}\phi \leq 2\pi$ and

$$\iint_R e^{(x^2+y^2)} x^2 dxdy = \int_0^1 \int_{\frac{3\pi}{2}}^{2\pi} \exp\left(r^2\right) r^3 \cos^2(\phi)\, dr d\phi$$

$$= \frac{\pi}{4}\frac{1}{2}\left[ye^y - e^y\right]_0^1 = \frac{\pi}{8}.$$

b. Let the surface S be the graph of the function $f(x, y) = \exp(x + y)$, where (x, y) satisfy

$$|x| + |y| \leq 2.$$

Find the surface integral

$$\iint_S z^2\, dS.$$

[Hint: Use the change of variables: $u = x + y$, $v = x - y$.]

If S is the graph of a function f over R, then

$$\iint_S g(x, y, z)\, dS = \iint_R g(x, y, f(x, y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy.$$

Also, in the coordinates u, v, the region R is defined by

$$-2 \leq u \leq 2, \quad -2 \leq v \leq 2.$$

At last, Jacobian of the above transformation, is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}.$$

Thus,

$$\iint_S z^2\, dS = \frac{1}{2}\int_{-2}^{2}\int_{-2}^{2} e^{2u}\sqrt{2e^{2u}+1}\, dudv = \frac{1}{2}\cdot 4 \cdot \frac{1}{6}\left[\left(2e^{2u}+1\right)^{3/2}\right]_{-2}^{2}$$

$$= \frac{1}{3}\left[\left(2e^4+1\right)^{3/2} - \left(2e^{-4}+1\right)^{3/2}\right].$$

3. a. State the Divergence Theorem carefully.

Let D be a bounded domain in $\mathcal{R}^3$ surrounded by a smooth surface S. Let

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k},$$

be a smooth vector-field in D. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dxdydz.$$

b. Let D be a cylinder,

$$x^2+y^2 \le 1, \quad 0 \le z \le 2.$$

Let $\mathbf{F}$ be a vector field

$$\mathbf{F}(x,y,z) = (1-a^2)x^3\mathbf{i} + (1-a^2)y^3\mathbf{j} + (x^2+y^2)z\mathbf{k},$$

where a is a real number. Find the flux of $\mathbf{F}$ through S, where S is the surface surrounding D.

We have

$$\nabla\cdot\mathbf{F} = 3(1-a^2)x^2 + 3(1-a^2)y^2 + (x^2+y^2) = (4-3a^2)(x^2+y^2).$$

By the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D (4-3a^2)(x^2+y^2)\,dxdydz.$$

Using the cylindrical coordinates, ρ, z, ϕ, we get

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = (4-3a^2)\int_0^2\int_0^{2\pi}\int_0^1 \rho^3 d\rho dz d\phi = (4-3a^2)\pi.$$

c. Let $\mathbf{F}$ and S be as in Part b. Find the values of a when the flux is equal to 0.

It is clear from the above that $a = \pm\sqrt{4/3}$.

d. Use the Divergence Theorem to prove the First Green's identity:

$$\iiint_V \left(f\,\nabla^2 g + \nabla f\cdot\nabla g\right)\,dxdydz = \iint_S f\frac{\partial g}{\partial \mathbf{n}}\,dS.$$

Here $f(x,y,z)$, $g(x,y,z)$ are smooth functions in a bounded domain $V \subset \mathcal{R}^3$, S is a smooth surface surrounding V and $\mathbf{n}$ is an outward-looking unit normal to V.

Let

$$\mathbf{F} = f\,\nabla g = f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}.$$

Then,

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right) =$$

$$f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right) = f\,\nabla^2 g + \nabla f \cdot \nabla g.$$

On the other hand,

$$\mathbf{F}\cdot\mathbf{n} = f\,(\nabla g \cdot \mathbf{n}) = f\frac{\partial g}{\partial \mathbf{n}}.$$

The result now follows from the Divergence Theorem.

4. a. State Stoke's Theorem carefully.

Given a curve C with the anti-clockwise direction and a capping surface S, so that C is the boundary of S, let $\mathbf{F}(x, y, z)$ be a smooth vector field defined on S. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS.$$

b. Verify Stoke's Theorem for the vector field

$$\mathbf{F}(x, y, z) = y\mathbf{i} + 2z\mathbf{j} + xz\mathbf{k}$$

and the surface S defined by

$$x^2 + y^2 + z^2 = 25, \quad z \geq 4.$$

i. The curve C is the circle $x^2 + y^2 = 9$ lying on the plane $z = 4$. Use parametrization

$$\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j} + 4\mathbf{k}, \quad \mathbf{r}'(t) = -3\sin(t)\mathbf{i} + 3\cos(t)\mathbf{j}.$$

Thus,

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -9\sin^2(t) + 24\cos(t),$$

so that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\left(-9\sin^2(t) + 24\cos(t)\right)\,dt = -9\pi.$$

ii. On the other hand,

$$\nabla \times \mathbf{F} = -2\mathbf{i} - z\mathbf{j} - \mathbf{k},$$

and S is the graph of the function $f(x, y) = \sqrt{25 - x^2 - y^2}$ with the domain $R = \{(x, y) : x^2 + y^2 \leq 9\}$. As

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{25 - x^2 - y^2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{25 - x^2 - y^2}},$$

we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R \left(\frac{2x}{\sqrt{25 - x^2 - y^2}} + \frac{2zy}{\sqrt{25 - x^2 - y^2}} - 1 \right) dxdy.$$

In polar coordinates, (r, ϕ),

$$\iint_R \left(\frac{-2x}{\sqrt{25 - x^2 - y^2}} + \frac{-2zy}{\sqrt{25 - x^2 - y^2}} - 1 \right) dxdy =$$

$$\int_0^3 \int_0^{2\pi} \left(\frac{-2r\cos(\phi)}{\sqrt{25 - r^2}} - 2r\sin(\phi) - 1 \right) r dr d\phi = -9\pi,$$

since $\int_0^{2\pi} \cos(\phi) d\phi = \int_0^{2\pi} \sin(\phi) d\phi = 0$.

5. a. State Green's Theorem in the plane carefully.

Let C be a smooth closed curve in the $xy-$plane oriented in the anti-cloackwise direction. Let R be a bounded region surrounded by C. Suppose that

$$\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$$

be a smooth vector field on R. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

b.) Sketch the closed curve C which is described as follows: Begin at point $(0, 0)$ and go to the point $(\pi, 0)$ along the straight line. Then go back to $(0, 0)$ along the curve described by the equation $y = \sin(x)$. This description also gives a correct orientation of C.

c. Let

$$\mathbf{F}(x,y) = y\mathbf{i} + (x^2y + \exp(y^2))\mathbf{j}.$$

Use Green's Theorem to calculate the circulation of $\mathbf{F}$ around C.

Solution (11 points):
We have $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2xy - 1$. Thus,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (2xy-1)\,dxdy = \int_0^\pi \left[\int_0^{\sin(x)} (2xy-1)dy\right] dx$$

$$= \int_0^\pi \left[x\sin^2(x) - \sin(x)\right] dx = \int_0^\pi \left[\frac{1}{2}x - \frac{1}{2}x\cos(2x) - \sin(x)\right] dx = \frac{\pi^2}{4} - 2.$$

6. a. Describe the necessary and sufficient conditions such that , for any $\mathbf{X_0}$, $\mathbf{X_1} \in \mathcal{R}^3$, the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the choice of C and depends only on $\mathbf{X_0}$ and $\mathbf{X_1}$.

Solution (5 points):
Since $\mathcal{R}^3$ is simply connected, $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on $\mathbf{X_0}$ and $\mathbf{X_1}$ iff $\nabla\times\mathbf{F} = \mathbf{0}$.

b. (unseen) Let $\mathbf{F} = \dfrac{2xz}{1+x^2z}\mathbf{i} + y\mathbf{j} + \dfrac{x^2}{1+x^2z}\mathbf{k}$.
Show that $\nabla\times\mathbf{F} = \mathbf{0}$ and find a potential function for $\mathbf{F}$.

Consider

$$\mathbf{E} = \nabla\times\mathbf{F} = E_1\mathbf{i} + E_2\mathbf{j} + E_3\mathbf{k}.$$

We have

$$E_1 = \frac{\partial}{\partial y}\left(\frac{x^2}{1+x^2z}\right) - \frac{\partial}{\partial z}y = 0;$$

$$E_1 = \frac{\partial}{\partial z}\left(\frac{2xz}{1+x^2z}\right) - \frac{\partial}{\partial x}\left(\frac{x^2}{1+x^2z}\right) = 0;$$

$$E_3 = \frac{\partial}{\partial x}y - \frac{\partial}{\partial y}\left(\frac{2xz}{1+x^2z}\right) = 0.$$

As $\mathcal{R}^3$ is simply connected, the function

$$f(x,y,z) = \int_C \mathbf{F}\cdot\mathbf{r},$$

where C is a curve from $\mathbf{0}$ to $\mathbf{X} = (\mathbf{x},\mathbf{y},\mathbf{z})$, is a potential function for $\mathbf{F}$.
Let C be a straight line,

$$\mathbf{r}(t) = xt\mathbf{i} + yt\mathbf{j} + zt\mathbf{k}, \quad \mathbf{r}'(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Then

$$\mathbf{F}\cdot\mathbf{r}' = \frac{2x^2zt^2}{1+x^2zt^3} + y^2t + \frac{x^2zt^2}{1+x^2zt^3} = \frac{3x^2zt^2}{1+x^2zt^3} + y^2t$$

Thus,

$$f(x,y,z) = \int_0^1 \left(\frac{3x^2zt^2}{1+x^2zt^3} + y^2t\right) dt = \ln(1+x^2z) + \frac{1}{2}y^2.$$

c. Let

$$\mathbf{G} = \frac{2xz}{1+x^2z}\mathbf{i} + (x+y)\mathbf{j} + \frac{x^2}{1+x^2z}\mathbf{k}.$$

Let C be a unit circle centered at $\mathbf{0}$ lying in the plane $y = z$.
Find

$$\oint_C \mathbf{G}\cdot d\mathbf{r}.$$

[Hint: Use Part (b).]

As $\mathbf{G} = \mathbf{F} + x\mathbf{j}$, due to Part (b),

$$\oint_C \mathbf{G}\cdot d\mathbf{r} = \oint_C x\mathbf{j}\cdot d\mathbf{r}.$$

The curve C may be parametrised by

$$x(t) = \cos(t),\ y(t) = z(t) = \frac{1}{\sqrt{2}}\sin(t); \quad x'(t) = -\sin(t),\ y'(t) = z'(t) = \frac{1}{\sqrt{2}}\cos(t).$$

Thus,

$$\oint_C \mathbf{G}\cdot d\mathbf{r} = \int_0^{2\pi} \cos(t)\frac{1}{\sqrt{2}}\cos(t)\,dt = \frac{\pi}{\sqrt{2}}$$